LOCALITY CRITERIA FOR COCOMMUTATIVE HOPF ALGEBRAS

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Abstract. We prove that a finite-dimensional cocommutative Hopf algebra H is local, if and only if the subalgebra generated by the first term of its coradical filtration H_1 is local. In particular if H is connected, H is local if and only if all the primitive elements of H are nilpotent.

1. Introduction

The structure theorem for cocommutative Hopf algebras is due to Milnor, Moore, Cartier and Kostant around 1963. Assume the base field is algebraically closed. In characteristic zero, it says that any cocommutative Hopf algebra is a smash product of the universal enveloping algebra of a Lie algebra and a group algebra. The statement becomes less definite in positive characteristic, where the universal enveloping algebra of a Lie algebra is replaced by any connected cocommutative Hopf algebra.

In this paper, we investigate one ring-theoretic property for finite-dimensional cocommutative Hopf algebras, i.e., the criteria for locality. We will prove the following main result:

Theorem A. Let H be a finite-dimensional cocommutative Hopf algebra over any arbitrary field k. Then H is local if and only if the subalgebra generated by H_1 is local.

We introduce the lower (upper) power series for (co)commutative Hopf algebras in positive characteristic in Section 3. Our approach is based on the duality theorem for those two series proved in Section 4. By using it, we verify a special case for Theorem A when the cocommutative Hopf algebra is connected in the next section 5. The Cartier-Kostant-Milnor-Moore theorem is reviewed in section 6 and the complete proof of Theorem A is given in section 7. We also state the corollary after the main result:

Corollary B. Let H be finite-dimensional connected Hopf algebra. Then the following are equivalent:

(i) H is local.
(ii) $u(\mathfrak{g})$ is local, where $\mathfrak{g}$ is the primitive space of H.
(iii) All the primitive elements of H are nilpotent.

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This corollary uses Engel's theorem in the representation theory of Lie algebras. Hence we can view our result as a generalization of Engel's theorem in the category of cocommutative Hopf algebras of finite dimension. In the last section, two examples are provided in order to show that the cocommutativity condition is necessary in Theorem A. The author does not know how to generalize this locality criteria to infinite-dimensional cocommutative Hopf algebras.

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2. PRELIMINARY

Throughout we work over a field k. For a finite-dimensional k-linear space V, identify V with its double dual V^{**} through the natural isomorphism. The standard collection $(H, m, u, \Delta, \varepsilon, S)$ is used to denote a Hopf algebra. We first recall some basic definitions and facts regarding H.

2.1. **Definition.** [3, Definitions 5.1.5, 5.2.1] The *coradical* H_0 of H is the sum of all simple subcoalgebras of H. The Hopf algebra H is *pointed* if every simple subcoalgebra is one-dimensional, and H is *connected* if H_0 is one-dimensional. For each $n \geq 1$, inductively set

$$H_n = \Delta^{-1}(H \otimes H_{n-1} + H_0 \otimes H).$$

The chain of subcoalgebras $H_0 \subseteq H_1 \subseteq \ldots \subseteq H_{n-1} \subseteq H_n \subseteq \ldots$ is the *coradical filtration* of H.

2.2. **Remark.** In the above definition of coradical filtration, it is same to define $H_n = \Delta^{-1}(H \otimes H_0 + H_{n-1} \otimes H)$ for each $n \geq 1$. In the following, assume H to be finite-dimensional.

(a) H^* has a natural Hopf algebra structure and $H = H^{**}$ under the natural isomorphism.
(b) Denote J as the Jacobson radical of H^*. Then $H_n = (H^*/J^{n+1})^*$ for any $n \geq 0$.
(c) H is local if and only if its augmented ideal $H^+ = \mathrm{Ker}\varepsilon$ is nilpotent if and only if H^* is connected.
(d) Let K be a Hopf subalgebra of H. Then $K_n \subseteq H_n$ for all $n \geq 0$. Moreover, if H is connected, so is K.

2.3. **Definition.** [3, Definitions 3.4.1, 3.4.5] A Hopf subalgebra K of H is *normal* if both

$$\sum (Sh_1)kh_2 \subseteq K \text{ and } \sum h_1k(Sh_2) \subseteq K,$$

for all $k \in K, h \in H$. A Hopf ideal I of H is *normal* if both

$$\sum h_1Sh_3 \otimes h_2 \subseteq H \otimes I \text{ and } \sum h_2 \otimes (Sh_1)h_3 \subseteq I \otimes H,$$

for all $h \in I$.

2.4. **Remark.** Let H be any Hopf algebra with Hopf subalgebra K and Hopf ideal I.

(a) If K is normal, then $K^+H = HK^+$ is a Hopf ideal of H.
(b) For finite-dimensional H, the Hopf ideal I is normal if and only if $(H/I)^*$ is a normal Hopf subalgebra of H^*.

Convention. Throughout, when K is a normal Hopf subalgebra of H, we use H/K to denote the quotient Hopf algebra H/HK^+.

2.5. **Lemma.** *Suppose* $\operatorname{char} k = p > 0$. *Let $K \subseteq H$ be finite-dimensional connected Hopf algebras. Then*

$$\dim H/\dim K \geq p^{\dim H_n/K_n},$$

for any n such that $K_i = H_i$ for all $i < n$.

Proof. It follows from [7, lemma 4.1], the inequality holds for the minimal integer n such that $K_n \neq H_n$. Then it is easy to check the statement. □

2.6. **Remark.** Let $K \subseteq H$ be finite-dimensional Hopf algebras. By [5, Corollary 5.6.4], H is isomorphic to $K \otimes H/K^+H$ as left K-modules and right H/K^+H-comodules. As a consequence, $\dim(H/K) = \dim H/\dim K$ provided that K is normal.

2.7. **Lemma.** *Let $K \subseteq L$ be normal Hopf subalgebras of a finite-dimensional Hopf algebra H. Then $(H/K)^*/(H/L)^* = (L/K)^*$. In particular, $H^*/(H/K)^* = K^*$ for any normal Hopf subalgebra K.*

Proof. The first result follows from [7, lemma 5.1]. For $H^*/(H/K)^* = K^*$, let $K = k$ and $L = K$ in the general case. □

2.8. **Lemma.** *Let H be a finite-dimensional Hopf algebra, and $E \supseteq k$ be a field extension. Then*

(i) *H is local if and only if any subalgebra of H is local.*
(ii) *H is local if and only if $H \otimes E$ is local.*
(iii) *$(H \otimes E)_n \subseteq H_n \otimes E$ for all $n \geq 0$*

Proof. (i) Suppose that H is local and A is any subalgebra of H. Denote J as the Jacobson radical of A and $I = H^+ \cap A$. By Remark 2.2(c), H^+ is nilpotent and so is $I \subseteq H^+$. Hence $I \subseteq J$. Moreover, $J \subseteq I$ for I has codimension one in A. Therefore $J = I$ and A is local. The inverse is trivial.

(ii) It is easy to see that $H \otimes E$ is a Hopf algebra [3, pp. 21] and $(H \otimes E)^+ = H^+ \otimes E$. Suppose that H is local. Then $(H \otimes E)^+$ is nilpotent for H^+ is. Hence $H \otimes K$ is local by Remark 2.2(c). Assume $H \otimes E$ to be local. Therefore $H^+ = H^+ \otimes 1 \subseteq H^+ \otimes E$ is a nilpotent ideal of H. So H is local.

(iii) Denote J and J' as the Jacobson radicals of H^* and $H^* \otimes E$. By Remark 2.2(b), the assertion is equivalent to that there is a surjection from $(H^*/J^n) \otimes E$ to $(H^* \otimes E)/J'^n$ for all $n \geq 0$. It is true because $J \subseteq J'$ for J is nilpotent. □

3. Upper and lower power series

Throughout this section assume that char $k = p > 0$.

3.1. **Definition.** Define the *lower power series* of a commutative Hopf algebra H as:

$$\Gamma_n(H) = \{h^{p^n} | h \in H\},$$

for all $n \geq 0$. Define the *upper power series* of a cocommutative Hopf algebra H as:

$$\Gamma^0(H) = k, \text{ and } \Gamma^n(H) = k\langle H_{p^{n-1}} \rangle$$

for $n \geq 1$, where $k\langle H_{p^{n-1}} \rangle$ denotes the subalgebra generated by the p^{n-1}-th term of its coradical filtration.

3.2. **Proposition.** *Regard to the above definitions, we have*

(i) *The lower power series of a commutative Hopf algebra is a descending chain of normal Hopf subalgebras.*
(ii) *The upper power series of a cocommutative Hopf algebra is an increasing chain of Hopf subalgebras.*

Proof. The statement with respect to the lower power series is easy to check by definition. Concerning (ii), the antipode S is an anticoalgebra map by [4, 4.0.1]. Morevover, it is a bijection by [**?**]. By the definition of coradical filtration and Remark 2.2, $SH_{p^{n-1}} = H_{p^{n-1}}$ for any $n \geq 1$. Notice that $\Gamma^n(H)$ is generated by $H_{p^{n-1}}$, which is a subcoalgebra. Then it becomes a Hopf subalgebra of H. □

3.3. **Remark.** Let H be a finite-dimensional cocommutative Hopf algebra. All $\Gamma_n(H^*)^+H^*$ are normal Hopf ideals of H^*. Moreover, $(H^*/\Gamma_n(H^*)^+H^*)^*$ are normal Hopf subalgebras of H by Remark 2.4(b).

4. Duality

In the following two sections, assume that k is algebraically closed with char $k = p > 0$. We want to prove the following duality theorem about lower and upper power series:

4.1. **Theorem.** *Let H be a finite-dimensional cocommutative connected Hopf algebra. Then*

$$\Gamma^n(H) = (H^*/\Gamma_n(H^*))^* \text{ and } \Gamma_n(H^*) = (H/\Gamma^n(H))^*.$$

First, we fix some notations in this section. Let H be a finite-dimensional cocommutative connected Hopf algebra. Therefore H^* is commutative local by Remark 2.2(c). We will write Γ^n for the upper powers series of H and Γ_n for the lower power series of H^*. Denote J_n as the Jacobson radical of Γ_n for each n, where specially write $J = J_0$. Since Γ_n is local by Lemma 2.8(i), it is clear that $J_n = \Gamma_n^+$ by Remark 2.2(c) and $H^*/\Gamma_n = H^*/J_nH^*$ for all $n \geq 0$.

4.2. **Lemma.** *We have $J_n \subseteq J^{p^n}$ and $J_n \bigcap J^{p^n+1} = J_n^2$. Moreover,*

$$\dim(\Gamma_{n-1}/\Gamma_n) = p^{\dim\left(J_{n-1}/(J^{p^{n-1}+1} \bigcap J_{n-1})\right)},$$

for any $n \geq 1$.

Proof. According to [6, Theorem 14.4], we can write

$$H^* = k\left[x_1, x_2, \cdots, x_d\right] \Big/ \left(x_1^{p^{e_1}}, x_2^{p^{e_2}}, \cdots, x_d^{p^{e_d}}\right)$$

as an algebra. Therefore each term Γ_n in the lower power series of H^* is generated by $\left\{x_i^{p^n} \middle| 1 \le i \le d\right\}$ and

$$J_n = \left(x_1^{p^n}, x_2^{p^n}, \cdots, x_d^{p^n}\right).$$

It follows that $J_n \subseteq J^{p^n}$. Hence $J_n^2 = J_n J_n \subseteq J^{p^n} J = J^{p^n+1}$ and $J_n^2 \subseteq J_n \bigcap J^{p^n+1}$. Then in order to show $J_n \bigcap J^{p^n+1} = J_n^2$, it suffices to prove that $\left(J_n \bigcap J^{p^n+1}\right)/J_n^2 = 0$. It is clear that any element in $\left(J_n \bigcap J^{p^n+1}\right)/J_n^2$ can be represented by $\sum \lambda_i x_i^{p^n}$ for $\lambda_i \in k$. Without loss of generality, we can assume that all $x_i^{p^n} \neq 0$. Observe that any term $\sum \alpha_i x_i^{d_i} \in J^{p^n+1}$ must have $d_i \geq p^n + 1$ or $\alpha_i = 0$. This implies that all $\lambda_i = 0$, which completes the proof. By the description of each Γ_n, there exists some integer l such that

$$\Gamma_{n-1}/\Gamma_n \cong k\left[y_1, \cdots, y_l\right] / \left(y_1^p, \cdots, y_l^p\right)$$

as algebras. Moreover, $l = \dim J_{n-1}/J_{n-1}^2$. Therefore

$$\dim\left(\Gamma_{n-1}/\Gamma_n\right) = p^l = p^{\dim\left(J_{n-1}/J_{n-1}^2\right)} = p^{\dim\left(J_{n-1}/(J^{p^{n-1}+1} \bigcap J_{n-1})\right)}.$$

□

4.3. **Lemma.** *We have for each $n \geq 1$*

$$p^{\dim\left((J_{n-1}H^* + J^{p^{n-1}+1})/J^{p^{n-1}+1}\right)} = \dim(H^*/\Gamma_n)^*/\dim(H^*/\Gamma_{n-1})^*.$$

Proof. We know $\Gamma_n \subseteq \Gamma_{n-1} \subseteq H^*$ are normal Hopf subalgebras of H^* by Proposition 3.2(i). Apply the isomorphism in Lemma 2.7, we have

$$\dim\left((H^*/\Gamma_n)^*/(H^*/\Gamma_{n-1})^*\right) = \dim\left(\Gamma_{n-1}/\Gamma_n\right).$$

Moreover, $\left(H^*/\Gamma_{n-1}\right)^*$ is normal in H hence in $\left(H^*/\Gamma_n\right)^*$ by Remark 3.3. Then by Remark 2.6, we have

$$\dim(H^*/\Gamma_n)^*/\dim(H^*/\Gamma_{n-1})^* = \dim\left((H^*/\Gamma_n)^*/(H^*/\Gamma_{n-1})^*\right) = \dim\left(\Gamma_{n-1}/\Gamma_n\right)$$

According to Lemma 4.2, $J_{n-1}J \subseteq J^{p^{n-1}}J = J^{p^{n-1}+1}$. Hence the natural map from J_{n-1} to $(J_{n-1}H^* + J^{p^{n-1}+1})/J^{p^{n-1}+1}$ is surjective, which has kernel $J_{n-1} \cap J^{p^{n-1}+1}$. Then the result follows by Lemma 4.2. □

4.4. **Lemma.** *We have $H_{p^n-1} \subseteq (H^*/\Gamma_n)^*$. Moverover, $\Gamma^n \subseteq (H^*/\Gamma_n)^*$ for all $n \geq 0$.*

Proof. There is a natural surjection $H^*/\Gamma_n = H^*/(J_nH^*) \twoheadrightarrow H^*/J^{p^n}$ since $J_n \subseteq J^{p^n}$ by Lemma 4.2. Notice that the dual of the last term is H_{p^n-1} in $H^{**} = H$ by Remark 2.2(b). Therefore we have $H_{p^n-1} \subseteq (H^*/\Gamma_n)^*$ by taking the dual of the above surjection. Because $p^{n-1} \leq p^n - 1$, $\Gamma^n(H) = k\langle H_{p^{n-1}}\rangle \subseteq k\langle H_{p^n-1}\rangle \subseteq (H^*/\Gamma_n)^*$. □

Proof of Theorem 4.1. First of all, we prove $\Gamma^n = (H^*/\Gamma_n)^*$ by inductions on n. When $n = 0$, both sides are the base field k. Next assume that the statement is true for $n-1$ and turn to the n case. Write $C = \Gamma^{n-1}$ and $D = \Gamma^n$, where $C = (H^*/\Gamma_{n-1})^*$ by induction. Consider $C \subseteq D \subseteq H$ as connected Hopf subalgebras of H by Remark 2.2(d). Moreover, by Lemma 4.4, $C_m = D_m = H_m$, whenever $m \leq p^{n-1} - 1$. Then we have

$$\dim D/\dim C \geq p^{\dim\left(D_{p^{n-1}}/C_{p^{n-1}}\right)}$$

by Lemma 2.5. According to Remark 2.2(b), a simple calculation yields that

$$\dim \frac{D_{p^{n-1}}}{C_{p^{n-1}}} \quad = \quad \dim \frac{H_{p^{n-1}}}{C_{p^{n-1}}} = \dim \frac{H^*/J^{p^{n-1}+1}}{H^*/(J_{n-1}H^* + J^{p^{n-1}+1})} = \dim \frac{J_{n-1}H^* + J^{p^{n-1}+1}}{J^{p^{n-1}+1}}.$$

Therefore by Lemma 4.3 and Lemma 4.4 , we have

$$\begin{aligned} \dim(H^*/\Gamma_n)^*/\dim(H^*/\Gamma_{n-1})^* \quad &\geq \quad \Gamma^n/\dim\Gamma^{n-1} = \dim D/\dim C \\ &\geq \quad p^{\dim\left(D_{p^{n-1}}/C_{p^{n-1}}\right)} = \dim(H^*/\Gamma_n)^*/\dim(H^*/\Gamma_{n-1})^*. \end{aligned}$$

Hence $\dim\Gamma_n = \dim(H^*/\Gamma_n)^*$ and so $\Gamma_n = (H^*/\Gamma_n)^*$. The other statement is checked as below. By Lemma 2.7,

$$(H/\Gamma^n)^* = (H^{**}/(H^*/\Gamma_n)^*)^* = (\Gamma_n)^{**} = \Gamma_n.$$

4.5. **Corollary.** *For the factors, we have*

$$\Gamma^n/\Gamma^m = (\Gamma_m/\Gamma_n)^*$$

for any $n \geq m$.

Proof. By Lemma 2.7, whenever $n \geq m$, we have

$$\Gamma^n/\Gamma^m = (H/\Gamma_n)^*/(H/\Gamma_m)^* = (\Gamma_m/\Gamma_n)^*.$$

□

4.6. **Corollary.** *The upper power series of H is a sequence of normal Hopf subalgebras.*

Proof. It follows from Theorem 4.1 and Remark 3.3. □

5. Finite-Dimensional Cocommutative Connected Hopf Algebras

We still assume k to be algebraically closed of characteristic $p > 0$. Recall that a *coalgebra filtration* of a coalgebra C is any set of subspaces $\{A_n\}_{n\geq 0}$ satisfying the two conditions below: (i) $A_n \subseteq A_{n+1}$, $C = \cup_{n\geq 0} A_n$ (ii) $\Delta A_n \subseteq \sum_{i=0}^n A_i \otimes A_{n-i}$. We state the following two dualized properties:

5.1. **Proposition.** *Let H be a finite-dimensional cocommutative connected Hopf algebra with upper power series $\{\Gamma^n(H)\}$. Then the following are equivalent:*

(i) $\Gamma^1(H)$ *is local.*
(ii) $\Gamma^n(H)/\Gamma^{n-1}(H)$ *is local for all $n \geq 1$.*
(iii) H *is local.*

5.2. **Proposition.** *Let K be a finite-dimensional commutative local Hopf algebra with lower power series $\{\Gamma_n(K)\}$. Then the following are equivalent:*

(i) *$H/\Gamma_1(K)$ is connected.*
(ii) *$\Gamma_{n-1}(K)/\Gamma_n(K)$ is connected for all $n \geq 1$.*
(iii) *K is connected.*

Proof. By Corollary 4.5 and Remark 2.2(c), those two statements are equivalent to each other condition by condition. Hence we can prove them together. The strategy is to show (i) $\Rightarrow$ (ii) in Proposition 5.2 and (ii) $\Rightarrow$ (iii) together with (iii) $\Rightarrow$ (i) in Proposition 5.1.

(i) $\Rightarrow$ (ii) in Proposition 5.2. Write $C = K/\Gamma_1(K)$ and $D = \Gamma_{n-1}(K)/\Gamma_n(K)$ for any $n \geq 1$. Define map $f : K \to \Gamma_{n-1}(K)$ as $f(x) = x^{p^{n-1}}$ for all $x \in K$. It is clear that f is surjective and it induces a surjection from C to D, which we still denote as f. Notice that f is not k-linear, but it is easy to show that that $\{f(C_n)\}_{n\geq 0}$ is a coalgebra filtration of D. Because C_n is the coradical filtration of C, by [3, Theorem 5.2.2], $\{C_n\}_{n\geq 0}$ is exhausted in C. Hence $D = \bigcup_{n\geq 0} f(C_n)$. Moreover since K is commutative:

$$\begin{aligned}\Delta\left(f(C_n)\right) &= \left(\Delta(C_n)\right)^{p^{n-1}} \subseteq \left(\sum_{i=0}^{n} C_i \otimes C_{n-i}\right)^{p^{n-1}} \subseteq \sum_{i=0}^{n} C_i^{p^{n-1}} \otimes C_{n-i}^{p^{n-1}} \\ &\subseteq \sum_{i=0}^{n} f\left(C_i\right) \otimes f\left(C_{n-i}\right)\end{aligned}$$

for all $n \geq 0$. By [3, Lemma 5.3.4], $D_0 \subseteq f(C_0)$ and the result follows.

(ii) $\Rightarrow$ (iii) in Proposition 5.1. Because H is finite-dimensional, there exists some integer d such that $\Gamma^d(H) = H$. Since each factor $\Gamma^n(H)/\Gamma^{n-1}(H)$ is local for all $1 \leq n \leq d$, therefore there are integers s_n such that $(\Gamma^n(H)^+)^{s_n} \subseteq \Gamma^n(H)\Gamma^{n-1}(H)^+$ for all $1 \leq n \leq d$. Denote $s = \prod s_n$, then $(H^+)^s = (\Gamma^d(H)^+)^s = 0$. Thus H is local. Finally (iii) $\Rightarrow$ (i) is from Lemma 2.8(i). $\square$

6. Pointed Cocommutative Hopf Algebras

Let H be any Hopf algebra, and G denote all the group-like elements of H. Recall [3, Definition 5.6.1] that any *irreducible component* of H is a maximal subcoalgebra, where any two of its nonzero subcoalgebras have nonzero intersection. We use H_g to denote the irreducible component of H containing g for every $g \in G$.

6.1. **Proposition.** [3, Corollary 5.6.4]

(i) *$H_x H_y = H_{xy}$ and $SH_x \subseteq H_{x^{-1}}$ for all $x, y \in G$.*
(ii) *H_e is a Hopf subalgebra of H, where e is the identity of G.*
(iii) *If H is pointed cocommutative, then $H_e \# kG \cong H$ via $h\#x \mapsto hx$.*

6.2. **Lemma.** *Let H be a finite-dimensional pointed cocommutative Hopf algebra. Then*

(i) *H_e is a connected normal Hopf subalgebra of H.*
(ii) *$H/H_e^+ H \cong k[G]$.*
(iii) *H is local if and only if H_e and $k[G]$ are local.*

Proof. (i) By Proposition 6.1(iii), we know H is generated by H_e and $k[G]$. We only need to check that for every $g \in G$, $gH_eg^{-1} \subseteq H_e$. It is true since $gH_eg^{-1} \subseteq H_gH_eH_{g^{-1}} = H_{geg^{-1}} = H_e$. We know H_e is connected for it contains only one simple subcoalgebra ke.

(ii) It follows from Proposition 6.1 such that $H \cong H_e \# kG$.

(iii) Suppose that H_e and $k[G]$ are local. Then there exist integers n, m such that $(H_e^+)^n = 0$ and $((G-1)k[G])^m = 0$. By part (ii), we have $(H^+)^m \subseteq H_e^+ H$. Hence $(H^+)^{nm} \subseteq (H_e^+ H)^n = 0$, which implies that H is local. The inverse direct comes from the Lemma 2.8(i). □

7. The Main Theorem

Let H be a finite-dimensional Hopf algebra over any arbitrary field k. We still use $\Gamma^1(H)$ to denote the subalgebra of H generated by H_1. Notice that $\Gamma^1(H)$ is again a Hopf subalgebra of H.

7.1. **Theorem.** *Let H be a finite-dimensional cocommutative Hopf algebra over any arbitrary field k. Then H is local if and only if the subalgebra generated by H_1 is local.*

Proof. Denote $K = H \otimes \bar{k}$. According to Lemma 2.8(ii), H is local if and only if K is local. Moreover by Lemma 2.8(iii), $\Gamma^1(K) \subseteq \Gamma^1(H) \otimes \bar{k}$ since $K_1 \subseteq H_1 \otimes \bar{k}$. Hence without loss of generality, we can assume k to be algebraically closed. Thus H is pointed by [3, pp. 76].

If k has characteristic zero, H must be a group algebra by [1, Theorem 1.1]. Hence H is local if and only if it is just the base field k. The statement is trivial. On the other hand, assume that char $k \neq 0$. Denote $A = H_e$ and $G = \mathrm{G}(H)$ as in Proposition 6.1. Since $k[G] \subseteq H_1 \subseteq \Gamma^1(H)$, it is local by Lemma 2.8(i). So is $\Gamma^1(A)$ since $A_1 \subseteq H_1$ by Remark 2.2(d). By Proposition 5.1, A is local for it is cocommutative and connected. Finally, the proof is complete with the help of Lemma 6.2(iii). □

7.2. **Corollary.** *Let H be finite-dimensional connected Hopf algebra. Then the following are equivalent:*

(i) *H is local.*
(ii) *$u(\mathfrak{g})$ is local, where $\mathfrak{g}$ is the primitive space of H.*
(iii) *All the primitive elements of H are nilpotent.*

Proof. In the connected case, the base field k has positive characteristic and $\Gamma^1(H)$ is the restricted enveloping algebra of the primitive space $\mathfrak{g}$. Hence the equivalence between (i) and (ii) follows from Theorem 7.1. (ii) $\Rightarrow$ (iii) is clear since $u(\mathfrak{g})$ contains all the primitive elements of H and its augmentation ideal is nilpotent. (iii) $\Rightarrow$ (ii). In characteristic p, $\mathfrak{g}$ is a restricted Lie algebra. (iii) asserts that all elements $x \in \mathfrak{g}$ are nilpotent. Then $(\mathrm{ad}x)^{p^n} = \mathrm{ad}(x^{p^n}) = 0$ for sufficient larger n. By Engel's Theorem [2, §3.2], $\mathfrak{g}$ is nilpotent. Any representation of $u(\mathfrak{g})$ is a restricted representation of $\mathfrak{g}$. Therefore any of its irreducible representation of is one-dimensional with trivial action of the augmentation ideal of $u(\mathfrak{g})$. Hence the augmentation ideal of $u(\mathfrak{g})$ is nilpotent and $u(\mathfrak{g})$ is local.

□

8. Examples

In the following suppose the base field k has characteristic $p \neq 0$. We give two parametric families of connected p^3-dimensional non-local Hopf algebras, where all the primitive elements are nilpotent. This ensures the necessity of cocommutativity in our main theorem.

8.1. **Example.** *Let A be the k-algebra generated by elements x, y, z subject to the following relations*

$$[x, y] = 0,\ [x, z] = \sigma x,\ [y, z] = (1 - \sigma)y,$$
$$x^p = 0,\ y^p = 0,\ z^p - z = \lambda x + \mu y.$$

Then A becomes a connected Hopf algebra via

$$\varepsilon(x) = 0,\ \Delta(x) = x \otimes 1 + 1 \otimes x,\ S(x) = -x,$$
$$\varepsilon(y) = 0,\ \Delta(y) = y \otimes 1 + 1 \otimes y,\ S(y) = -y,$$
$$\varepsilon(z) = 0,\ \Delta(z) = z \otimes 1 + 1 \otimes z + x \otimes y,\ S(z) = -z + xy,$$

where $\sigma, \lambda, \mu \in k$ satisfying $\sigma^p = \sigma, \lambda\sigma = (1 - \sigma)\mu = 0$. We denote it as $A(\sigma, \lambda, \mu)$.

8.2. **Example.** *Let B be the k-algebra generated by elements x, y, z subject to the following relations*

$$[x, y] = 0,\ [x, z] = x + \sigma y,\ [y, z] = 0,$$
$$x^p = y,\ y^p = 0,\ z^p = z,$$

where $\sigma \in k$. Then B becomes a connected Hopf algebra via

$$\varepsilon(x) = 0,\ \Delta(x) = x \otimes 1 + 1 \otimes x,\ S(x) = -x,$$
$$\varepsilon(y) = 0,\ \Delta(y) = y \otimes 1 + 1 \otimes y,\ S(y) = -y,$$
$$\varepsilon(z) = 0,\ \Delta(z) = z \otimes 1 + 1 \otimes z + (x + \sigma y) \otimes y,\ S(z) = -z + (x + \sigma y)y.$$

We denote it as $B(\sigma)$.

References

[1] N. Andruskiewitsch, About finite dimensional Hopf algebras. *Contemp. Math.*, 294(2002), 1-57.
[2] J.E. Humphreys, Introduction to Lie algebras and representation theory, vol. 9, Springer,1980.
[3] S. Montgomery, Hopf Algebras and Their Actions on Rings, *Amer. Math. Soc.*, 82(1993).
[4] M.E. Sweedler, Hopf Algebras, Benjamin, New York, 1969.
[5] H.J. Schneider, Normal basis and transitivity of crossed products for Hopf algebras, *J. Algebra* 152 (1992), 289-312.
[6] W.C. Waterhouse, Introduction to affine group schemes, vol. 66, Springer, 1979.
[7] X. Wang, Connected Hopf algebras of dimension p^2, submitted, arXiv:1208.2280.

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